Handwritten Notes

On

Theory of Equations

M. **Equations, Inequations & Expressions.**

* **Linear equ^n:** $ax + b = 0$, $a, b \in R$, $a \neq 0$.

$$\text{root (unique)} = -\frac{b}{a}.$$

* **Quadratic equ^n:** $ax^2 + bx + c = 0$, $a, b, c \in R$, $a \neq 0$.

$$\text{roots} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \Big| \quad b^2 - 4ac = D = \text{discriminant}.$$

* **Quadratic equ^n – nature of roots:**

1. If $a, b, c \in R$, $a \neq 0$, then a) If $D < 0$, then non-real complex roots, b) If $D > 0$ then real, distinct roots. $ax^2 + bx + c = a(x - x_1)(x - x_2)$.
c) If $D = 0$, then real & equal roots
2. D is perfect square of a rational number, then roots are rational; not a perfect square then are irrational.
3. If one root is $p + iq$ then the other $p - iq$.
4. If one root be $p + \sqrt{q}$ other will be $p - \sqrt{q}$.
5. If $a = 1$, $b, c \in I$ & the roots are rational, then they must be integers.
6. If more than two roots then $a = b = c = 0$

* **Relation between roots & coefficients:**

If α & β be the roots,
i) $\alpha + \beta = -b/a$ ii) $\alpha\beta = c/a$ iii) $(x - \alpha)(x - \beta) = 0$ or

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0.$$

* **Quadratic expression:** $(ax^2 + bx + c)$ $a, b, c \in R$, $a \neq 0$.

$$f(x) = ax^2 + bx + c.$$

* Condition for common roots: $ax^2+bx+c=0$, $a'x^2+b'x+c'=0$ | $a \cdot a' \neq 0$, $\frac{a}{a'} \neq \frac{b}{b'}$

- one root common – $(bc'-b'c)(ab'-a'b) = (ca'-c'a)^2$
- two roots common – $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$

* Graph of a quadratic expression: $f(x) = ax^2+bx+c = y$. $(a \neq 0)$.

$$\left(y+\frac{D}{4a}\right) = a\left(x+\frac{b}{2a}\right)^2 \quad \Bigg| \quad y+\frac{D}{4a} = Y,\ x+\frac{b}{2a} = X$$

$$Y = aX^2 \Rightarrow X^2 = \frac{1}{a} \cdot Y$$

1. $y = f(x)$ is parabolic.
2. axis of parabola $X = 0$, $x = -\frac{b}{2a}$.
3. If $a > 0$, the parabola opens upwards; $a < 0$, opens downwards.
4. i) Intersection with x axis –

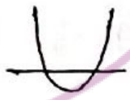
$$ax^2+bx+c = 0 \Rightarrow x = \frac{-b \pm \sqrt{D}}{2a}.$$

for I) $D = 0$, $x = -b/2a$


$a>0, D=0$

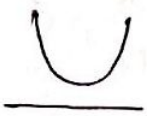

$a<0, D=0$.

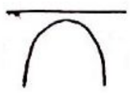
II) $D > 0$, $\alpha, \beta = \frac{-b \pm \sqrt{D}}{2a}$.


$a>0, D>0$


$a<0, D>0$.

III) $D < 0$, $\alpha, \beta = \frac{-b \pm \sqrt{D}}{2a}$ [imaginary, doesn't cut x axis].


$a>0, D<0$


$a<0, D<0$.

ii) Intersection with y axis – $y = c$.

5. Vertex of parabola $= \left(-\frac{b}{2a}, -\frac{D}{4a}\right)$.

for $a > 0$, then $f(x)$ has least value at $x = -\frac{b}{2a}$,

$f\left(-\frac{b}{2a}\right) = -\frac{D}{4a}$.

for $a<0$, then $f(x)$ has greatest value at

$x = -b/2a$, $f\left(-\frac{b}{2a}\right) = -\frac{D}{4a}$.

* Sign of Quadratic Expression:

1. $a > 0, D < 0$ — $f(x) > 0 \ \forall x \in R$.
2. $a < 0, D < 0$ — $f(x) < 0 \ \forall x \in R$
3. $a > 0, D = 0$ — $f(x) > 0 \ \forall x$ except vertex
4. $a < 0, D = 0$ — $f(x) < 0 \ \forall x$ except vertex
5. $a > 0, D > 0$ — $f(x) > 0 \ \forall x \in (-\infty, \alpha) \cup (\beta, \infty)$
   $[\alpha < \beta]$ $f(x) < 0 \ \forall x \in (\alpha, \beta)$
6. $a < 0, D > 0$ — $f(x) < 0 \ \forall x \in (-\infty, \alpha) \cup (\beta, \infty)$
   $f(x) > 0 \ \forall x \in (\alpha, \beta)$.

* Location of roots:

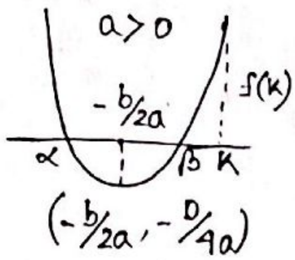
1. Conditions for a number k (If both roots are less than k).

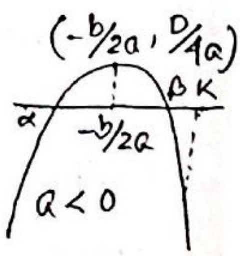
i) $D \geq 0$

ii) $a f(k) > 0$

iii) $k > -\frac{b}{2a}$

where $\alpha \leq \beta$.





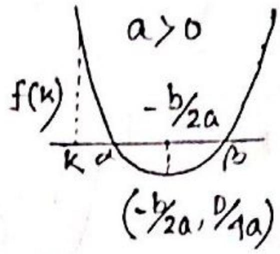
2. Condition for a number k (both roots $> k$).

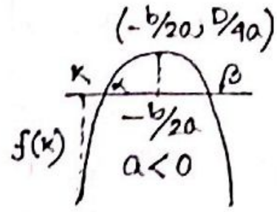
i) $D \geq 0$.

ii) $af(k) > 0$.

iii) $k < -b/2a$

where $\alpha \leq \beta$.



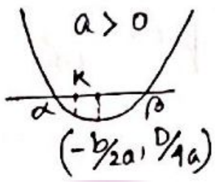


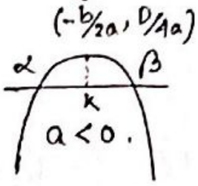
3. Condition for a number k ($\alpha < k < \beta$).

i) $D > 0$

ii) $af(k) < 0$

where $\alpha < \beta$.



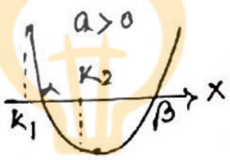


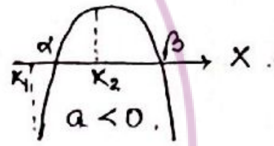
4. Condition for numbers k_1 & k_2 (If one root lies in the interval (k_1, k_2)):

i) $D > 0$

ii) $f(k_1) \cdot f(k_2) < 0$

where $\alpha < \beta$.





5. Condition for numbers k_1 & k_2 (If both roots confined between k_1 & k_2):

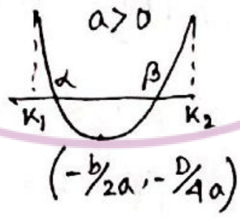
i) $D \geq 0$.

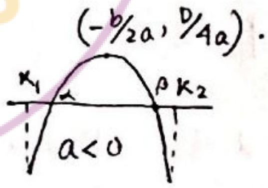
ii) $af(k_1) > 0$

iii) $af(k_2) > 0$.

iv) $k_1 < -b/2a < k_2$

where $\alpha < \beta$, $k_1 < k_2$





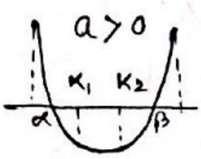
6. Condition for numbers k_1 & k_2 (If k_1, k_2 lie between the roots).

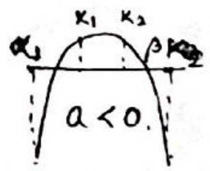
i) $D > 0$

ii) $af(k_1) < 0$.

iii) $af(k_2) < 0$

where $\alpha < \beta$.





* **Wavy Curve Method:** $f(x) = (x-a_1)^{k_1}(x-a_2)^{k_2}\cdots(x-a_n)^{k_n}$

$$a_1 < a_2 < a_3 < \cdots < a_{n-1} < a_n$$

$$f(x) = \frac{(x-a_1)^{n_1}(x-a_2)^{n_2}\cdots(x-a_k)^{n_k}}{(x-b_1)^{m_1}(x-b_2)^{m_2}\cdots(x-b_p)^{m_p}}.$$

Marking the numbers $a_1, a_2, \ldots, a_n$ on the real axis, and plus sign in the interval of the right of the largest of these numbers, ie, on the right of a_n. If k_n is even then we put plus sign on the left of a_n. When passing through the point a_{n-1} the polynomial changes ~~sing~~ sign if k_{n-1} is an odd number, $f(x)$ has same sign if k_{n-1} is an even number.

* **Equations of higher degree:**

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0. \qquad (a_0 \neq 0)$$

If $\alpha_1, \alpha_2, \ldots, \alpha_n$ be n roots, then.

$$\sum \alpha_1\alpha_2\cdots\alpha_p = (-1)^p \frac{a_p}{a_0}.$$

* **Trinomial Equations:** $ax^{2n} + bx^n + c = 0$, $a \neq 0$, $n \geq 2$

for $n=2$ the trinomial is called a biquadratic equation.

* Equ^{n}s which can be reduced to linear, quadratic & biquadratic equ^{n}s:

I. $(x-a)(x-b)(x-c)(x-d) = A$.

when $a<b<c<d$, $b-a = d-c$, can be solved by change of variable.

$$y = \frac{(x-a)+(x-b)+(x-c)+(x-d)}{4}$$

$$y = x - \frac{a+b+c+d}{4}.$$

II. $(x-a)(x-b)(x-c)(x-d) = Ax^2$. when $ab = cd$, can be reduced to a collection of two quadratic equations by a change of variable $y = x + \frac{ab}{x}$.

III. $(x-a)^4 + (x-b)^4 = A$ by substitution

$$y = \frac{(x-a)+(x-b)}{2}.$$

* Rational Equations: $\frac{P(x)}{Q(x)} = 0$. $P(x)$ & $Q(x)$ are polynomials, $Q(x) \neq 0$.

• Equ^{n}s of the form.

i) $\frac{Ax}{ax^2+b_1x+c} \pm \frac{Bx}{ax^2+b_2x+c} = C$. ii) $\frac{ax^2+b_1x+c}{ax^2+b_2x+c} \pm \frac{ax^2+b_3x+c}{ax^2+b_4x+c} = A$ $(ac \neq 0)$

iii) $\frac{ax^2+b_1x+c}{ax^2+b_2x+c} = \frac{Ax}{ax^2+b_3x+c}$ $(ac \neq 0,\ A \neq 0)$.

Divide above & below by x in each fraction on LHS, then putting $ax + \frac{c}{x} = t$. then (i), (ii) (iii) become $\frac{A}{t+b_1} \pm \frac{B}{t+b_2} = C$; $\frac{t+b_1}{t+b_2} \pm \frac{t+b_3}{t+b_4} = A$;

$$\frac{t+b_1}{t+b_2} = \frac{A}{t+b_3}.$$

* Equations containing absolute values:

$|x| = x$, if $x \geq 0$.

$|x| = -x$, if $x < 0$.

form 1: $f(|x|) = g(x)$. is equivalent to the systems. $f(x) = g(x)$ if $x \geq 0$

$f(-x) = g(x)$ if $x < 0$.

form 2: $|f(x)| = g(x) \equiv \begin{cases} f(x) = g(x) , \text{ if } g(x) \geq 0 \\ -f(x) = g(x) , \text{ if } g(x) \geq 0. \end{cases}$

form 3: $h(|f(x)|) = g(x). \equiv \begin{cases} h(f(x)) = g(x), & f(x) \geq 0 \\ h(-f(x)) = g(x) , & f(x) < 0 \end{cases}$

form 4: $|f(x) + g(x)| = |f(x)| + |g(x)| \equiv$

$f(x)g(x) \geq 0.$

form 5: $|f_1(x)| + |f_2(x)| \cdots + |f_n(x)| = g(x).$

first find all critical points, then graph start with +ve sign if coefficient of x is +ve.