Handwritten Notes

On

The Straight Line

The Straight Line. (2D)

* The general equation of the first degree, in x, y, i.e. $ax + by + c = 0$ represents a straight line.
* To completely determine the equation of a straight line, we require two conditions.
* Slope of a line: If inclination is θ, then $\tan\theta$ is called the slope or gradient of the line.

If (x_1, y_1) & (x_2, y_2) are two points on a line, then the slope m is $(y_2 - y_1)/(x_2 - x_1)$. $(x_1 \neq x_2)$

- If three points A, B, C are collinear, then $m(AB) = m(BC) = m(CA)$.

* Angle between two lines: The acute angle θ between the lines having slopes m_1 & m_2 is given by

$$\theta = \tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|.$$

- If two lines are parallel, $m_1 = m_2$ & if perpendicular $m_1 m_2 = -1$.

* Lines parallel to co-ordinate axes:
  i) to y axis: $x = a$ (equ^n of y axis $x = 0$)
  ii) to x axis: $y = b$ (equ^n of x axis $y = 0$).

* Intercepts of a line on axes: x intercept a, y intercept b

quad I — $+a, +b$	quad III - $-a, -b$
quad II - $-a, +b$	quad IV - $+a, -b$

* Different forms of equ^n of st. line:

1. Slope-intercept form: Equ^n of st. line whose slope is m & which cuts an intercept c on y-axis, is

$$y = mx + c.$$

2. Point-slope form of line: Equ^n of line that passes through (x_1, y_1) & has slope m is $y - y_1 = m(x - x_1)$.

3. Two-point form: Equ^n of line passing through two given points (x_1, y_1) & (x_2, y_2) is $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$.

$$\text{or} \quad \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

4. The Intercept form: Equ^n of line which cuts off intercepts of length of a & b on x axis & y axis respectively, is $\frac{x}{a} + \frac{y}{b} = 1$. or $\begin{vmatrix} a & 0 & 1 \\ x & y & 1 \\ 0 & b & 1 \end{vmatrix} = 0.$

5. Normal/Perpendicular form: Equ^n of line upon which the length of perpendicular from origin is p & normal makes an angle α with +ve direction of x-axis is

$$x\cos\alpha + y\sin\alpha = p.$$

[p is always +ve]

$$\text{or} \quad \begin{vmatrix} p\sec\alpha & 0 & 1 \\ x & y & 1 \\ 0 & p\,\mathrm{cosec}\,\alpha & 1 \end{vmatrix} = 0.$$

6. Symmetric/Parametric Form: Equ^n of st. line passing through (x_1, y_1) ~~(x_2, y_2)~~ & making an angle θ with the +ve direction of x- axis is

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r.$$

, where r is the directed distance between the points (x, y) & (x_1, y_1).

- $\left.\begin{array}{l} x = x_1 + r\cos\theta \\ y = y_1 + r\sin\theta \end{array}\right\}$ parametric equ^{n}s of st. line

* Reduction of general equation to standard form:

1. Slope-intercept form: $y = \left(-\frac{A}{B}\right)x + \left(-\frac{C}{B}\right)$. $[B \neq 0]$

   Slope, $m = -\frac{A}{B}$.

   y intercept, $c = -\frac{C}{B}$.

- cor. Angle between $A_1x + B_1y + C_1 = 0$ & $A_2x + B_2y + C_2 = 0$.

$$\tan\theta = \left|\frac{A_1B_2 - A_2B_1}{A_1A_2 + B_1B_2}\right|.$$

If lines are parallel $\frac{A_1}{A_2} = \frac{B_1}{B_2}$.

If perpendicular $A_1A_2 + B_1B_2 = 0$.

- If two lines are coincident $\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$

2. Intercept form : $\dfrac{x}{-C/A} + \dfrac{y}{-C/B} = 1$. $[A, B \neq 0]$

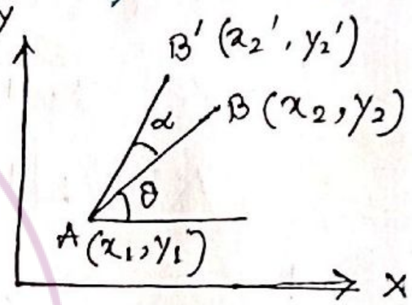
x intercept $= -C/A$.

y intercept $= -C/B$.

3. Normal form : $\left(-\dfrac{A}{\sqrt{A^2+B^2}}\right)x + \left(-\dfrac{B}{\sqrt{A^2+B^2}}\right)y = \dfrac{C}{\sqrt{A^2+B^2}}$.

* New Co-ordinates when rotated by an angle α :

$$\frac{x_2 - x_1}{\cos\theta} = \frac{y_2 - y_1}{\sin\theta} = r = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

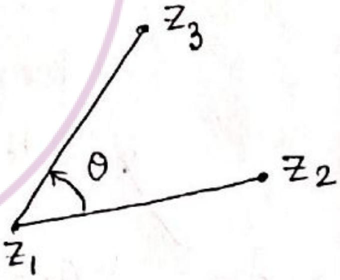


$x_2 = x_1 + r\cos\theta$

$y_2 = y_1 + r\sin\theta$.

$x_2' = x_1 + r\cos(\theta+\alpha)$; $y_2' = y_1 + r\sin(\theta+\alpha)$.

* Complex number as a rotating arrow in Argand Plane :



$$z_3 - z_1 = (z_2 - z_1)e^{i\theta}.$$

$$\Rightarrow \frac{z_3 - z_1}{z_2 - z_1} = e^{i\theta}.$$

* Position of a point relative to a line:

The point $(x_1 y_1)$ is on one side or the other side of the line $ax + by + c = 0$ $(b > 0)$ according as $ax_1 + by_1 + c > 0$ or < 0.

* Position of two points relative to a given line: The points $P(x_1, y_1)$ & $Q(x_2, y_2)$ lie on the same or opposite sides of the line

$ax+by+c=0$ according as,

$$\frac{ax_1+by_1+c}{ax_2+by_2+c} > 0 \text{ or } < 0.$$

- Cor. A point (α, β) will lie on origin side of the line $ax+by+c=0$, if $a\alpha+b\beta+c$ and c have same sign.

* Equations of lines parallel and perpendicular to a given line: 1. Equ^n of line parallel to $ax+by+c=0$ is $ax+by+\lambda=0$, where λ is some constant. 2. Equ^n of line perpendicular to $ax+by+c=0$ is $bx-ay+\lambda=0$, where λ is some constant. Cor. Equ^n of line parallel to $ax+by+c=0$ & passing through (x_1, y_1) is $a(x-x_1)+b(y-y_1)=0$. Similarly, for perpendicular line $b(x-x_1)-a(y-y_1)=0$

*Distance of a point from a line:

Length of perpendicular from a point (x_1, y_1) to the line $ax+by+c=0$ is $\dfrac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$.

*Distance between two parallel lines:

Perpendicular distance between the lines $ax_1+by_1+c=0$ & $ax_1+by_1+c_1=0$ is $\dfrac{\lambda}{\sqrt{a^2+b^2}}$.

i) $\lambda = |c_1 - c|$, if both lines on same side of origin.

ii) $\lambda = |c_1| + |c|$, if they are on opposite sides, of origin.

* Area of Parallelogram: Area of parallelogram ABCD whose sides AB, BC, CD, DA are represented by $a_i x + b_i y + c_i = 0$.

$$\frac{P_1 P_2}{\sin\theta} \quad \text{or} \quad \left| \frac{|c_1 - d_1||c_2 - d_2|}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \right|$$

$a_1 x + b_1 y + c_1 = 0$
$a_2 x + b_2 y + c_2 = 0$
$a_1 x + b_1 y + d_1 = 0$
$a_2 x + b_2 y + d_2 = 0$

where, P_1 & P_2 are the distances between parallel sides and θ is the angle between two adjacent sides.

- cor. If $P_1 = P_2$ then the area of rhombus,

$$P_1^2/\sin\theta. = \frac{(c_1 - d_1)^2}{|a_1 b_2 - a_2 b_1| \sqrt{\frac{a_1^2 + b_1^2}{a_2^2 + b_2^2}}}$$

cor. If d_1 & d_2 are the lengths of two perpendicular diagonals of a rhombus then area of rhombus $= \frac{1}{2} d_1 d_2$.

cor. Area of parallelogram whose sides are $y = mx + a$, $y = mx + b$, $y = nx + c$ & $y = nx + d$ is $\left| \frac{(a-b)(c-d)}{(m-n)} \right|$.

* Concurrent lines: Three lines $a_i x + b_i y + c_i = 0$ are concurrent if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$.

or,

Condition for the lines $P = 0$, $Q = 0$, $Z = 0$ to be concurrent is that three constants l, m, n (not all zeroes at the same time) can be obtained such that $lP + mQ + nZ = 0$.

Straight Line

* Family of lines: Any line through the point of intersection of the lines $a_1x+b_1y+c_1=0$ & $a_2x+b_2y+c_2=0$ can be represented by

$$(a_1x+b_1y+c_1)+\lambda(a_2x+b_2y+c_2)=0.$$

where λ is a parameter which depends on the other property of line.

* The equations of the straight lines which pass through a given point (x_1, y_1) & make an angle α with the given straight line $y=mx+c$ are

$$y-y_1=\tan(\theta\pm\alpha)(x-x_1) \quad [m=\tan\theta].$$

* A line equally inclined with two lines: If two lines with slopes m_1 & m_2 be equally inclined to a line with slope m, then

$$\frac{m_1-m}{1+mm_1}=-\frac{m_2-m}{1+mm_2}.$$

* Equation of the bisectors: Equ^n of the bisectors of the angles between the lines $a_1x+b_1y+c_1=0$ & $a_2x+b_2y+c_2=0$ are given by,

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=\pm\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

Equ^n of bisector containing origin $\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$

Equ^n of " not containing origin $\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=-\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$

* Equation of bisector of the angle between the two lines containing points (h,k) will be,

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \pm \frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

according as $a_1h+b_1k+c_1$ & $a_2h+b_2k+c_2$ are of the same sign or opposite sign.

* Distinguishing the acute & obtuse angle bisectors : Equ^n of bisector

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \pm \frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}.$$

Condition	Acute angle bisector	Obtuse angle bisector
$a_1a_2+b_1b_2>0$	$-$	$+$
$a_2a_1+b_1b_2<0$	$+$	$-$.

* Foot of perpendicular drawn from the point (x_1,y_1) to the line $ax+by+c=0$:

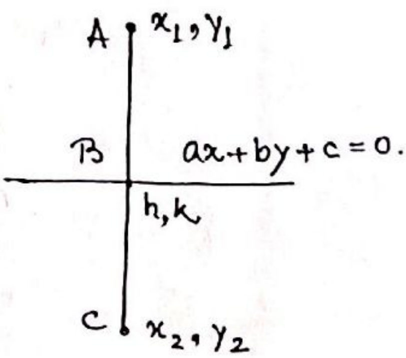
$$x = \frac{ac-b(bx_1-ay_1)}{b^2-a^2}$$

$$y = \frac{bc-a(bx_1-ay_1)}{b^2-a^2}.$$

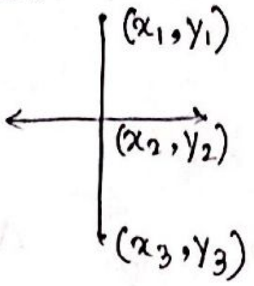
* Image or reflection of a point (x_1,y_1) about a line mirror :

If $B : h,k$ then

$$h = \frac{x_1+x_2}{2}, \quad k = \frac{y_1+y_2}{2}.$$



* Short-cut:



$$\frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = -\frac{ax_1 + by_1 + c}{(a^2 + b^2)}$$

$$\frac{x_3 - x_1}{a} = \frac{y_3 - y_1}{b} = -\frac{2(ax_1 + by_1 + c)}{a^2 + b^2}.$$

* Position of a point which lies inside a triangle: First draw the exact diagram of the problem. If the point $P(x_1, y_1)$ move on the line $y = ax + b$ for all x_1, y_1, then $P \equiv (x_1, ax_1 + b)$ & a portion DE of the line $y = ax + b$ (excluding D & E) lies within the triangle. Now, $y = ax + b$ cuts any two sides out of three sides, then find co-ordinates of D & E.

$$D \equiv (\alpha, \beta), \quad E \equiv (\gamma, \partial)$$

then $\alpha < x_1 < \gamma$ & $\beta < ax_1 + b < \partial$.

* If $A \equiv (x_1, y_1)$, $B \equiv (x_2, y_2)$ & $C \equiv (x_3, y_3)$ are the vertices of ΔABC, then $\angle A$ is acute or obtuse according as,

$$(x_1 - x_2)(x_1 - x_3) + (y_1 - y_2)(y_1 - y_3) > 0 \text{ or } < 0.$$

for $\angle B$,

$$(x_2 - x_3)(x_2 - x_1) + (y_2 - y_3)(y_2 - y_1) > 0 \text{ or } < 0$$

for $\angle C$,

$$(x_3 - x_1)(x_3 - x_2) + (y_3 - y_1)(y_3 - y_2) > 0 \text{ or } < 0.$$

* If the origin lies in the acute or obtuse angle between the lines $a_1x+b_1y+c_1=0$ & $a_2x+b_2y+c_2=0$ according as.

$$(a_1a_2+b_1b_2)\,c_1c_2 < 0 \text{ or } > 0.$$

* If (x_i, y_i) are the vertices of ΔABC then equations of right bisectors of side BC, CA, AB are

$$y(y_2-y_3)+x(x_2-x_3)=\left(\frac{x_2^2-x_3^2}{2}\right)+\left(\frac{y_2^2-y_3^2}{2}\right).$$

$$y(y_3-y_1)+x(x_3-x_1)=\left(\frac{x_3^2-x_1^2}{2}\right)+\left(\frac{y_3^2-y_1^2}{2}\right).$$

$$y(y_1-y_2)+x(x_1-x_2)=\left(\frac{x_1^2-x_2^2}{2}\right)+\left(\frac{y_1^2-y_2^2}{2}\right).$$

- equ^{n}s of medians AD, BE, CF are

$$y(x_2+x_3-2x_1)-x(y_2+y_3-2y_1)=y_1(x_2+x_3)-x_1(y_2+y_3).$$

$$y(x_3+x_1-2x_2)-x(y_3+y_1-2y_2)=y_2(x_3+x_1)-x_2(y_3+y_1).$$

$$y(x_1+x_2-2x_3)-x(y_1+y_2-2y_3)=y_3(x_1+x_2)-x_3(y_1+y_2).$$

- equ^{n}s of altitudes AL, BM, CN are

$$y(y_2-y_3)+x(x_2-x_3)=y_1(y_2-y_3)+x_1(x_2-x_3).$$

$$y(y_3-y_1)+x(x_3-x_1)=y_2(y_3-y_1)+x_2(x_3-x_1)$$

$$y(y_1-y_2)+x(x_1-x_2)=y_3(x_1-x_2)+x_3(x_1-x_2).$$

* If of ΔABC, BC: $a_1x+b_1y+c_1=0$, CA: $a_2x+b_2y+c_2=0$, AB: $a_3x+b_3y+c_3=0$, then

$|BC| : |CA| : |AB| =$

$$\sqrt{a_1^2+b_1^2}\left|\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}\right| : \sqrt{a_2^2+b_2^2}\left|\begin{vmatrix} a_3 & b_3 \\ a_1 & b_1 \end{vmatrix}\right| : \sqrt{a_3^2+b_3^2}\left|\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\right|$$

* Area of Triangle having corner points (x_1, y_1), (x_2, y_2), (x_3, y_3) $= \frac{1}{2}\left[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right]$

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

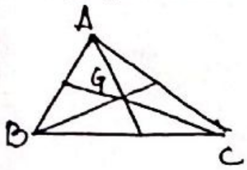
* In case of polygon with vertices (x_i, y_i), then

$$\text{area} = \frac{1}{2}\left|(x_1y_2 - y_1x_2) + (x_2y_3 - y_2x_3) + \dots + (x_{n-1}y_n - y_{n-1}x_n) + (x_ny_1 - y_nx_1)\right|.$$

* Centres connected with a triangle:

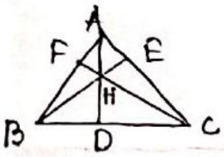
$A \equiv (x_1, y_1)$
$B \equiv (x_2, y_2)$
$C \equiv (x_3, y_3)$
$BC = a$, $CA = b$
$AB = c$

1. Centroid: Point of concurrency of the medians.



$$G \equiv \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right).$$

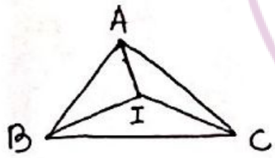
2. Orthocentre: Point of concurrency of the altitudes.



$$H \equiv \left(\frac{x_1 \tan A + x_2 \tan B + x_3 \tan C}{\tan A + \tan B + \tan C}, \frac{y_1 \tan A + y_2 \tan B + y_3 \tan C}{\tan A + \tan B + \tan C}\right).$$

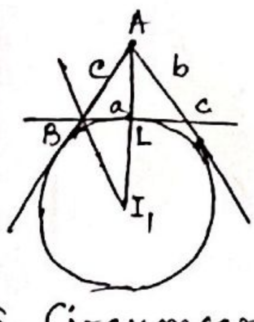
(NB - ΔDEF is orthic or pedal triangle).

3. Incentre: Point of concurrency of the internal bisectors of angles.



$$I \equiv \left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right).$$

4. Excentre: Point of concurrency of two external bisectors of angle.

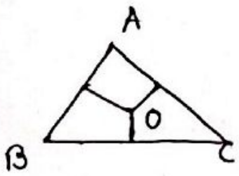


$$\frac{BL}{LC} = \frac{c}{b} \qquad \frac{AI_1}{I_1L} = -\frac{b+c}{a}$$

$$I_1 = \left(\frac{-ax_1 + bx_2 + cx_3}{-a+b+c}, \frac{-ay_1 + by_2 + cy_3}{-a+b+c}\right).$$

Similarly, I_2, I_3.

5. Circumcentre: Point of concurrency of perpendicular bisectors of sides.



$$O \equiv \left(\frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}\right)$$