Handwritten Notes

On

Sequences and Series

Sequences & Series.

* **Sequence:** A sequence is a function of natural numbers with codomain is the set of Real numbers or complex numbers.

A mapping $f: N \to C$ then $f(n) = t_n, n \in N$ is called a sequence to be denoted it by $\{f(1), f(2), \ldots, f(n)\} = \{t_1, t_2, \ldots, t_n\} = \{t_n\}$.

* **Series:** By adding or subtracting the terms of a sequence then we get a series.

* **Sequence by recursive relation:**

ie. $a_{n+1} = a_n + a_{n-1}$,

$$a_{n+1}^2 = a_n a_{n+2} + (-1)^n.$$

* **Arithmetic Sequence:** A sequence whose terms increase or decrease by a fixed number.

If $\{t_1, t_2, t_3, \ldots\}$ is such that $t_n - t_{n-1} =$ constant $\forall n \in N$, then its an arithmetic sequence.

- n^{th} term of AP: $T_n = a + (n-1)d. = S_n - S_{n-1}$
- n^{th} term from last: $T_n' = l - (n-1)d.$ [$l \to$ last term].
- Sum of first n terms:

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$= \frac{n}{2}[a + l]$$

$a \to$ first term

$d \to$ common term/difference.

* Imp. points on AP:.

i) If $a_1, a_2, a_3, \ldots$ are in AP, then

a) $a_1 \pm K, a_2 \pm K, a_3 \pm K$ are also in AP.

b) a_1K, a_2K, a_3K are also in AP.

ii) If $a_1, a_2, \ldots$ & $b_1, b_2, \ldots$ are two APs, then

a) $a_1 \pm b_1, a_2 \pm b_2, \ldots$ are in AP.

b) $a_1b_1, a_2b_2, \ldots$ aren't in AP.

iii) $a_r = \dfrac{a_{r-k} + a_{r+k}}{2} \quad \forall K, \quad 0 \le K \le n-r.$

iv) $2r+1$ numbers in AP can be taken as

$a-rd, a-(r-1)d, \ldots, a-d, a, a+d, \ldots, a+(r-1)d, a+rd$

v) $2r$ numbers in AP can be taken as

$[a-(2r-1)d, a-(2r-3)d, \ldots, a-3d, a-d, a+d,$
$a+3d, \ldots, a+(2r-3)d, a+(2r-1)d].$

vi) If n^{th} term of any sequence is linear expression in n $(an+b)$ the sequence is AP.

vii) If sum of n terms of any sequence is quadratic expression in n, (an^2+bn+c) then sequence is an AP.

* Geometric Sequence: A sequence where the ratio of any term and its just preceding term is constant.

Thus, $t_1, t_2, t_3, \ldots$ are in GP, then common ratio $r = \dfrac{t_n}{t_{n-1}}$.

- nth term of GP: $T_n = ar^{n-1} = l$ (last term).
- nth term from end: $\dfrac{l}{r^{n-1}}$.
- Sum of first n terms: $S_n = \begin{cases} \dfrac{a(r^n-1)}{r-1} = \dfrac{lr-a}{r-1}, & r \neq 1 \\ na, & r = 1. \end{cases}$
- Sum of infinite GP, when $|r|<1$,

when $n \to \infty$, $x^n \to 0$, $S_\infty = \dfrac{a}{1-r}$ $(|r|<1)$.

- finding the value of recurring decimal:

$R = 0.XYYY\ldots$

$\Rightarrow R \times 10^x = X.YYY\ldots$

$\Rightarrow R \times 10^{x+y} = XY.YYY\ldots$

$R = \dfrac{XY - X}{10^{x+y} - 10^x}$

If $R = 0.YYY$

$R = \dfrac{Y}{10^y - 1}$.

* Imp. points on GP:

i) If $a_1, a_2, \ldots$ are in GP, then

a) $a_1 k, a_2 k, a_3 k$, are in GP.

b) $1/a_1, 1/a_2, 1/a_3, \ldots$ are in GP.

c) $\log a_1, \log a_2, \log a_3, \ldots$ are in AP.

d) $a_1^r, a_2^r, a_3^r, \ldots$ are in GP.

e) $a_1 a_n = a_2 a_{n-1} = a_3 a_{n-2} = \ldots$

f) $a_r = \sqrt{a_{r-k} a_{r+k}}$ $\quad 0 \leq k \leq n-r$

g) $\dfrac{a_2}{a_1} = \dfrac{a_3}{a_2} = \ldots = \dfrac{a_n}{a_{n-1}}$

ii) If $a_1, a_2, a_3, \ldots$ & $b_1, b_2, b_3, \ldots$ are two GPs then, a) $a_1b_1, a_2b_2, a_3b_3, \ldots$ & $a_1/b_1, a_2/b_2, \ldots$ are in GP.

b) $a_1 \pm b_1, a_2 \pm b_2, \ldots$ are not in GP.

iii) $(2m+1)$ numbers in GP can be written as

$$\frac{a}{r^m}, \frac{a}{r^{m-1}}, \ldots, \frac{a}{r}, a, ar, \ldots, ar^{m-1}, ar^m$$

iv) $2m$ numbers in GP can be written as

$$\frac{a}{r^{2m-1}}, \frac{a}{r^{2m-3}}, \ldots, \frac{a}{r^3}, \frac{a}{r}, ar, ar^3, \ldots, ar^{2m-3}, ar^{2m-1}$$

* Harmonic Sequence: A sequence where the reciprocals of its terms are in AP. If $a_1, a_2, \ldots, a_n$ are in AP, $1/a_1, 1/a_2, \ldots, 1/a_n$ are in HP.

- n^{th} term, $T_n = \dfrac{1}{\frac{1}{a_1} + (n-1)\left(\frac{1}{a_2} - \frac{1}{a_1}\right)} = \dfrac{a_1 a_2}{a_2 + (n-1)(a_1 - a_2)}$.
- n^{th} term from end, $T_n' = \dfrac{a_1 a_2 a_n}{a_1 a_2 - a_n (n-1)(a_1 - a_2)}$.
- no term of HP can be zero, & there's no general formula for finding out the sum of n terms of HP.
- If $x_1, x_2, \ldots, x_n$ are in HP, then

$$x_1x_2 + x_2x_3 + \ldots + x_{n-1}x_n = (n-1)x_1x_n$$

- If every term of a HP is multiplied or divided by some non zero fixed quantity, the resulting progression is HP.

* Recognition of AP, GP & HP: If a, b, c are 3 successive terms of a sequence. Then,

AP – when $\frac{a-b}{b-c} = \frac{a}{a}$

GP – when $\frac{a-b}{b-c} = \frac{a}{b}$

HP – when $\frac{a-b}{b-c} = \frac{a}{c}$.

* Means:

a. Arithmetic Mean: If a, b, c are in AP, b is the AM of a & c.

i) Single AM of n positive integers:

$$AM = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

ii) n AMs between two numbers:

$$a, A_1, A_2, \ldots, A_n, b.$$

$$A_n = a + n\left(\frac{b-a}{n+1}\right).$$

b. Geometric Mean: If a, b, c are in GP, b is the GM of a & c.

i) Single GM of n +ve numbers:

$$GM = (a_1 a_2 a_3 \cdots a_n)^{1/n}$$

ii) n GMs between 2 numbers:

$$a, G_1, G_2, G_3, \ldots, G_n, b.$$

$$G_n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

C. Harmonic Mean: If a, b, c are in HP, b is the HM of a & c.

i) Single HM of n +ve integers:

$$\frac{1}{HM} = \frac{1}{n}\left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}\right).$$

ii) n HMs between two numbers:

$$a, H_1, H_2, \dots, H_n, b.$$

$$\frac{1}{H_n} = \frac{1}{a} + \frac{n(a-b)}{(n+1)ab}.$$

* Relation among AM, GM & HM.

→ For two +ve real numbers a & b.

$$A = \frac{a+b}{2}, \quad G = \sqrt{ab}, \quad H = \frac{2ab}{a+b}.$$

$$\boxed{AH = G^2} \qquad \boxed{A \geqslant G \geqslant H}$$

* If $a, A_1, A_2, \dots, A_n, b$ are in AP, then
$A_1 + A_2 + A_3 + \dots + A_n = n\,(\text{AM of } a \,\&\, b) = n \cdot \frac{a+b}{2}.$
* If $a, G_1, G_2, \dots, G_n, b$ are in GP, then
$G_1 \cdot G_2\, G_3 \dots G_n = (\text{GM of } a \,\&\, b)^n = (\sqrt{ab})^n.$
* If $a, H_1, H_2, \dots, H_n, b$ are in HP, then

$$\frac{1}{H_1} + \frac{1}{H_2} + \dots + \frac{1}{H_n} = n\left(\frac{1}{\text{HM of } a \,\&\, b}\right) = n\left(\frac{1}{\frac{2ab}{a+b}}\right).$$

* If three +ve unequal quantities a, b, c be in HP, then $a^n + c^n > 2b^n$, $n \in N$.

* Sigma Notation: i) $\sum_{i=1}^{m} a = am$

ii) $\sum_{i=1}^{K} a f(i) = a \sum_{i=1}^{K} f(i)$, iii) $\sum_{i=1}^{K} \{ f(i) \pm g(i) \} = \sum_{i=1}^{K} f(i) \pm \sum_{i=1}^{K} g(i)$.

* Important results:

a) $\Sigma n = \frac{n}{2}(n+1)$, b) $\Sigma n^2 = \frac{n(n+1)(2n+1)}{6}$

c) $\Sigma n^3 = \left\{ \frac{n(n+1)}{2} \right\}^2$.

* Arithmetico - Geometric Series: (AGS).

A series is said to be an arithmetico-geometric series if its each term is formed by multiplying the corresponding term of an AP & a GP. eg. $1 + 3x + 5x^2 + 7x^3 + \cdots$

- Sum of n terms of an AGS:

* $S_n = \frac{a}{1-r} + \frac{dr(1-r^{n-1})}{1-r} - \frac{[a+(n-1)d]r^n}{1-r}$.

of the AGS $a, (a+d)r, (a+2d)r^2, \ldots, [a+(n-1)d]r^{n-1}$.

When Sum to infinity $\left[|r|<1, \lim_{n\to\infty} r^n = 0 \right]$.

* $S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$.

* Method of difference: If the differences of the successive terms of a series are in AP or GP, we can find nth term of the series by the steps –

I) Denote the n^{th} term & the sum of the series upto n terms of the series by T_n & S_n respectively.

II) Rewrite the given series with each term shifted by one plane to the right.

III) Subtracting the above two forms of the series, find T_n.

* $v_n - v_{n-1}$ method: Let $T_1, T_2, T_3, \ldots$ be the terms of a sequence, if there exists a sequence $v_1, v_2, v_3, \ldots$ satisfying $T_K = v_K - v_{K-1}$, $K \geq 1$, then $S_n = \sum_{K=1}^{n} T_K = \sum_{K=1}^{n} (v_K - v_{K-1}) = v_n - v_0$.

* Weighted Means: Let $a_1, a_2, \ldots, a_n$ be positive real numbers & $w_1, w_2, \ldots, w_n$ be rational numbers. Then we define weighted arithmetic mean (A^*), weighted geometric mean (G^*) & weighted harmonic mean (H^*) as.

$$A^* = \frac{a_1 w_1 + a_2 w_2 + \cdots + a_n w_n}{w_1 + w_2 + \cdots + w_n}$$

$$G^* = \left(a_1^{w_1} \cdot a_2^{w_2} \cdots a_n^{w_n}\right)^{\frac{1}{w_1 + w_2 + \cdots + w_n}}$$

$$H^* = \frac{w_1 + w_2 + \cdots + w_n}{\frac{w_1}{a_1} + \frac{w_2}{a_2} + \cdots + \frac{w_n}{a_n}}$$

$A^* \geq G^* \geq H^*$ [Equality holds at either place iff $a_1 = a_2 = \cdots = a_n$.

* Cauchy's Schwartz Inequality: If $a_1, a_2, \ldots, a_n$ & $b_1, b_2, \ldots, b_n$ are $2n$ real numbers, then

$$(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2 \leq (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2)$$

with the equality holding iff $a_1/b_1 = a_2/b_2 = \cdots = a_n/b_n$