Handwritten Notes

On

Limit, Continuity and Differentiability

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Limits, Continuity & Differentiability.

* **Definition of Limit:** $f(x)$ approaches the limit L as x approaches x_0, if for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x, $0 < |x - x_0| < \delta$ & $|f(x) - L| < \varepsilon$.

* **Left & Right Hand Limit:** $\lim\limits_{x \to a^-} f(x) = L_l$, if for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x satisfying $a - \delta < x < a \Rightarrow |f(x) - L_l| < \varepsilon$ & it's the LH limit.

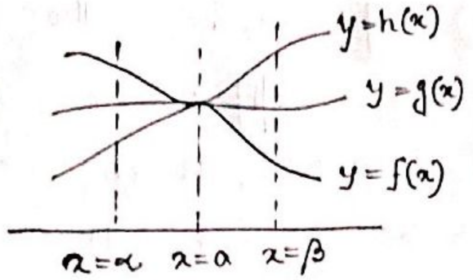
$$\lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h) \quad [x = a-h]$$

$\lim\limits_{x \to a^+} f(x) = L_R$, if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x satisfying $a < x < a + \delta \Rightarrow |f(x) - L_R| < \varepsilon$ & it's the RH limit. $\left[\lim\limits_{h \to 0} f(a+h)\right]$

* **Algebra of Limits:** $\lim\limits_{x \to a} f(x) = l$ & $\lim\limits_{x \to a} g(x) = m$.

1. $\lim\limits_{x \to a} \{f(x) \pm g(x)\} = l \pm m$
2. $\lim\limits_{x \to a} \{f(x) \cdot g(x)\} = l \cdot m$
3. $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{l}{m}$, $m \neq 0$.
4. $\lim\limits_{x \to a} K f(x) = K \lim\limits_{x \to a} f(x)$.
5. $\lim\limits_{x \to a} \{f(x) + K\} = \lim\limits_{x \to a} f(x) + K$
6. If $f(x) \leq g(x)$, for $x \in (a-h, a+h) - \{a\}$, for some $h > 0$, then $\lim\limits_{x \to a} f(x) \leq \lim\limits_{x \to a} g(x)$.
7. $\lim\limits_{x \to a} f\{g(x)\} = f\left\{\lim\limits_{x \to a} g(x)\right\}$.

* Squeeze Play Theorem (Sandwich Theorem):



If $f(x) \le g(x) \le h(x) \;\forall\; x \in (\alpha,\beta)-\{a\}$

& $\lim_{x\to a} f(x) = l = \lim_{x\to a} h(x)$ then

$\lim_{x\to a} g(x) = l$ where $a \in (\alpha,\beta)$.

* $\frac{0}{0}, \frac{\infty}{\infty}, \infty-\infty, 1^\infty, 0^0, \infty^0, \infty\times 0, \infty\times\infty$ are all indeterminate forms. eg. $\lim_{x\to a} \frac{x^n-a^n}{x-a} = \frac{0}{0}$, $\lim_{x\to 0}\frac{\sin x}{x} = \frac{0}{0}$.

* Important Expansions.

1. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots = \lim_{n\to\infty}\sum_{r=1}^{n}(-1)^{r+1}\frac{x^r}{r}$; $-1 < x \le 1$

2. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \lim_{n\to\infty}\sum_{r=0}^{n}\frac{x^r}{r!}$

3. $a^x = 1 + (x\ln a) + \frac{(x\ln a)^2}{2!} + \frac{(x\ln a)^3}{3!} + \dots$

4. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$ 5. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$

6. $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \dots$

7. $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots$ $-1 < x < 1$.
[n is rational or integral].

* Standard limits:

1. $\lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}$ 2. $\lim_{x\to 0}\frac{(1+x)^n-1}{x} = n$

3. $$\lim_{x\to\infty}\frac{a_0x^m + a_1x^{m-1}+\dots}{b_0x^n + b_1x^{n-1}+\dots} = \begin{cases} \frac{a_0}{b_0}, & \text{when } m=n \\ 0, & \text{when } m<n \\ \infty, & \text{when } m>n, \frac{a_0}{b_0}>0 \\ -\infty, & \text{when } m>n, \frac{a_0}{b_0}<0 \end{cases}$$

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4. $\lim_{x\to 0} \sin x = \lim_{x\to 0} \tan x = 0.$

5. $\lim_{x\to 0} \cos x = \lim_{x\to 0} \frac{1}{\cos x} = 1.$

6. $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{x}{\sin x} = \lim_{x\to 0} \frac{\sin^{-1}x}{x} = \lim_{x\to 0} \frac{x}{\sin^{-1}x} =$

$\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{x}{\tan x} = \lim_{x\to 0} \frac{\tan^{-1}x}{x} = \lim_{x\to 0} \frac{x}{\tan^{-1}x} = 1.$

7. $\lim_{x\to \infty} \frac{\sin x}{x} = \lim_{x\to\infty} \frac{\cos x}{x} = 0.$ [x measured in radians].

8. $$\lim_{x\to\infty} a^x = \begin{cases} 0, & 0 \le a < 1 \\ 1, & a = 1 \\ \infty, & a > 1 \\ \text{not exists}, & a < 0. \end{cases}$$

9. $\lim_{x\to 0} \frac{e^x - 1}{x} = 1.$, $\lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a \quad (a > 0)$

10. $\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$, $\lim_{x\to 0} \frac{\log_a(1+x)}{x} = \log_a e.$

11. $\lim_{x\to\infty} \left(1 + \frac{a}{x}\right)^x = \lim_{x\to 0} (1+x)^{a/x} = e^a$

$\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x\to 0} (1+x)^{1/x} = e.$

12. If $\lim_{x\to a} f(x)$ exists & positive then

$$\lim_{x\to a} [f(x)]^{f(x)} = e^{\lim_{x\to a} f(x)\cdot \ln[f(x)]}.$$

* L'Hospital's rule: If $\lim_{x\to a} \frac{f(x)}{g(x)}$, reduces to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ then differentiate Nr & Dr till this form is removed.

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{f''(x)}{g''(x)} = \cdots$$

* Standard Methods to find limits:

1. Factorization, 2. Rationalization, 3. By applying Standard limits, 4. Expansion of functions, 5. Substitution, 6. Use of logarithm.

* Continuity of a function at a point:

A function $f(x)$ is said to be continuous at $x = a$, if $\lim_{x \to a-} f(x) = \lim_{x \to a+} f(x) = f(a)$.

If not continuous at $x = a$, $f(x)$ is discontinuous at $x = a$.

- All polynomials, trigonometric functions, exponential and logarithmic functions are continuous in their domain.

- Reasons of discontinuity: 1. $\lim_{x \to a} f(x)$ does not exist, 2. $\lim_{x \to c} f(x) \neq f(c)$, 3. $f(x)$ is not defined at $x = b$.

- Types of discontinuity:

1. Removable: If $\lim_{x \to a} f(x)$ exists but $\lim_{x \to a+} f(x) \neq f(a)$ or iff $\lim_{x \to a-} f(x) = \lim_{x \to a+} f(x) \neq f(a)$.

2. Non-removable: i) If $\lim_{x \to a-} f(x) \neq \lim_{x \to a+} f(x)$

ii) If $\lim_{x \to a-} f(x)$, $\lim_{x \to a+} f(x)$ or both do not exist.

* Algebra of Continuous functions: $f(x)$ & $g(x)$ continuous at $x = a$.

1. $c f(x)$ is continuous at $x = a$.
2. $f(x) \pm g(x)$ is continuous at $x = a$
3. $f(x) \cdot g(x)$ & $f(x) / g(x)$ are continuous at $x = a$, provided $g(a) \neq 0$.

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Limits, Continuity & Differentiability.

- If $f(x)$ is continuous & $g(x)$ is discontinuous at $x = a$, then the product function $\phi(x) = f(x) \cdot g(x)$ is not necessarily be discontinuous at $x = a$.
- If $f(x)$ & $g(x)$ both are discontinuous at $x = a$ then the product function $\phi(x) = f(x) \cdot g(x)$ is not necessarily be discontinuous at $x = a$.
- If the function $y = f(x)$ is defined, continuous & strictly monotonic on the interval X, then there exists a single valued function $x = \varphi(y)$ defined, continuous & also strictly monotonic on the range of $y = f(x)$.
- If f is continuous at $x = c$ & g is continuous at $x = f(c)$ then the composite $g\{f(x)\}$ is continuous at $x = c$.

* Applications of Continuity:

1. Intermediate Value Theorem: $f(x)$ is continuous on interval I, & a & b are any two points of I. Then if y_0 is a number between $f(a)$ & $f(b)$, there exists a number c between a & b such that $f(c) = y_0$.

2. Extreme Value Theorem: If f is continuous at every point of a closed interval I, then f assumes both an absolute maximum value M & an absolute minimum value m somewhere in I.

* **Definition of Derivative:** The derivative $f'(x)$ of a function $y=f(x)$ at a given point x is defined as

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

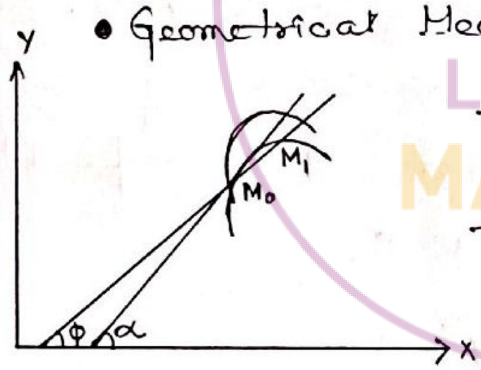
If this limit exists finitely then then the function $f(x)$ is called differentiable at the point x.

Lt derivative, $$f'_{-}(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Rt derivative, $$f'_{+}(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

- $f'(x)$ exists if $f'_{+}(x)$ & $f'_{-}(x)$ exist & $f'_{+}(x) = f'_{-}(x)$.

- Geometrical Meaning:



$$\tan\phi = \frac{\Delta y}{\Delta x}$$

$$\tan\alpha = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}.$$

Slope of the tangent to curve $y=f(x)$.

* **Differentiability on an interval:** $f(x)$ is differentiable on an interval if it has a derivative at each point of the interval. It is differentiable on a closed interval $[a,b]$ & if the limits $\lim_{h\to 0+} \frac{f(a+h)-f(a)}{h} =$ finite, $\lim_{h \to 0+} \frac{f(b)-f(b-h)}{h} =$ finite.

* Causes of Non-derivability: i) A corner, where function is continuous but LHD & RHD are finite but different.
ii) A cusp, where the slope approaches ∞ from one side & $-\infty$ from the other. iii) a vertical tangent where the slope approaches ∞ from both sides or approaches $-\infty$ from both sides.
iv) a discontinuity.

* Relation between derivability & continuity:

1. If $f'(a)$ exists then $f(x)$ is derivable at $x=a$ $\Rightarrow f(x)$ is continuous at $x=a$. In general, a function f is derivable at x then f is continuous at x, i.e. if $f(x)$ is derivable for every point of its domain of definition, then it is continuous in ~~that~~ domain. The converse is need not to be true.

2. Let $f'_+(a) = \lambda$ & $f'_-(a) = \mu$, $\lambda, \mu \to$ finite

i) $\lambda = \mu \Rightarrow f$ is derivable at $x=a$
$\Rightarrow f$ is continuous at $x=a$.

ii) $\lambda \neq \mu \Rightarrow f$ is not derivable at $x=a$ but f is continuous at $x=a$. If f is not differentiable but is continuous at $x=a$.

iii) If f is not continuous at $x=a$ then it is not differentiable at $x=a$.

* **Algebra of Derivatives:** If u & v are derivable functions of x, then

i) $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$

ii) $\frac{d}{dx}(Ku) = K\frac{du}{dx}$

iii) $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

iv) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\left(\frac{du}{dx}\right) - u\left(\frac{dv}{dx}\right)}{v^2}$

v) Chain rule. $y = f(u)$, $u = g(x)$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

vi) $(fog)(x) = f\{g(x)\} \Rightarrow (fog)'(x) = f'\{g(x)\} \cdot g'(x)$.

* **Derivative of $f(x)$ from first principle:**

If $f(x)$ is a derivable function, then

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{dy}{dx}.$$

* **Standard Formulae of Differentiation:**

1. $\frac{d}{dx}\sin x = \cos x$
2. $\frac{d}{dx}\cos x = -\sin x$
3. $\frac{d}{dx}\sec x = \sec x \tan x$
4. $\frac{d}{dx}\tan x = \sec^2 x$
5. $\frac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x \cot x$
6. $\frac{d}{dx}\cot x = -\operatorname{cosec}^2 x$.
7. $\frac{d}{dx}(\text{cons.}) = 0$
8. $\frac{d}{dx}x^n = nx^{n-1}$
9. $\frac{d}{dx}a^x = a^x \ln a$.
10. $\frac{d}{dx}e^x = e^x$
11. $\frac{d}{dx}\left(\log_a |x|\right) = \frac{1}{x}\log_a e$.
12. $\frac{d}{dx}\ln|x| = \frac{1}{x}$
13. $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$, $-1 < x < 1$.
14. $\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$, $-1 < x < 1$
15. $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$, $x \in R$
16. $\frac{d}{dx}\cot^{-1}x = \frac{-1}{1+x^2}$, $x \in R$.
17. $\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}$, $|x| > 1$
18. $\frac{d}{dx}\operatorname{cosec}^{-1}x = \frac{-1}{|x|\sqrt{x^2-1}}$, $|x| > 1$.

* **Differentiation of Implicit Function:** Differentiating each term of $\phi(x,y)=0$ w.r.t. x regarding y as a function of x & then collecting terms in $\frac{dy}{dx}$ together on one side. In answers of $\frac{dy}{dx}$ in the case of implicit functions, both x & y are present.

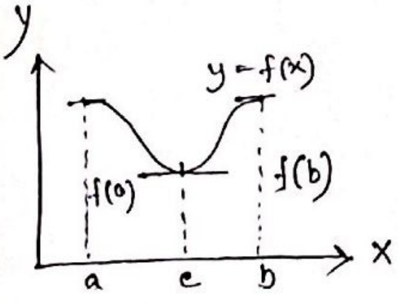
* Logarithmic function: If $y = u^v$, where u & v are the functions of x, then $\log y = v \log u$. Differentiating wrt x,

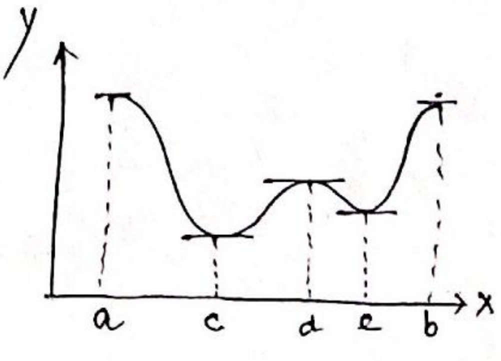
$$\frac{d}{dx}(u^v) = u^v\left[\frac{v}{u}\frac{du}{dx} + \log u \frac{dv}{dx}\right].$$

* Parametric function: If $x = f(t)$ & $y = g(t)$ then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)} \quad f'(t) \neq 0.$$

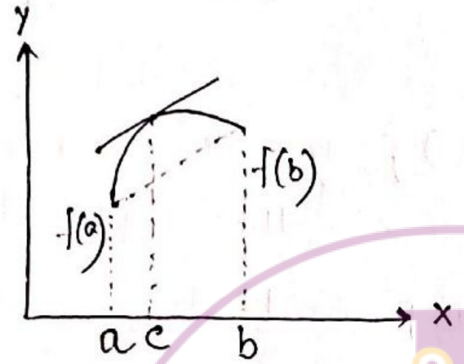
* Rolle's Theorem: If a function $f(x)$ is i) continuous in the closed interval $[a,b]$ ie continuous at each point in $[a,b]$ ii) differentiable in an open interval (a,b), iii) $f(a)=f(b)$, then, there will be at least one point c, in the interval (a,b) such that $f'(c)=0$.





* Lagrange's Mean Value Theorem: If a function $f(x)$ is i) continuous in the closed interval, ii) differentiable in the open interval (a, b) then

there will be at least one point c, where $a < c < b$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$



* Polynomial, exponential, sine, cosine functions are everywhere continuous & differentiable.

* Logarithmic function is continuous & differentiable in its domain.

* $\tan x$ is not continuous and differentiable at $x = \pm \pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots$

* $|x|$ is not differentiable at $x = 0$.

* If $f'(x)$ tends to $\pm\infty$ as $x \to k$, then $f(x)$ is not differentiable at $x = k$.