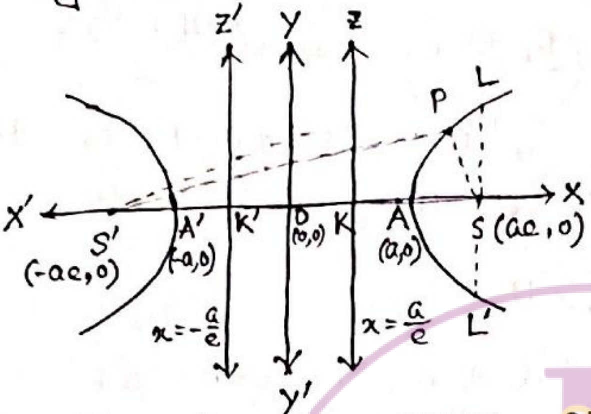
Handwritten Notes

On

Hyperbola

Hyperbola

1. Locus of a point whose ratio of the distance from a point and from a fixed line is always greater than unity. Particular case of conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ where $abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$ and $h^2 > ab$ (rectangular when $a+b=0, \Delta \neq 0$).

2. 

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$[b^2 = a^2(e^2-1)].$$

* $e = \sqrt{1 + \frac{(\text{con. axis})^2}{(\text{tran. axis})^2}}$

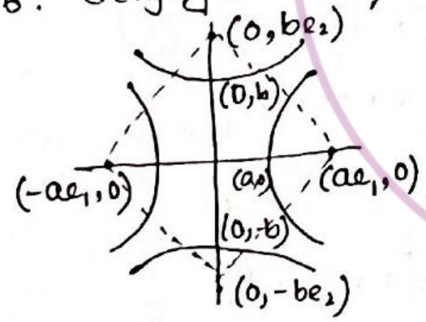
* $K, K' \rightarrow$ foot of directrix.

3. Length of Transverse axis $= 2a$
Length of Conjugate axis (imaginary) $= 2b$.

4. Latus rectum: (LSL'). length $= 2\frac{b^2}{a} = 2a(e^2-1)$.

5. Focal distance of a point: length $= 2a =$ length of transverse axis.

6. Conjugate hyperbola:



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

$$\Rightarrow \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1.$$

$$e_1^{-2} + e_2^{-2} = 1$$

The foci of the hyperbola & its conjugate are concyclic & form vertices of a square. (common asymptotes)

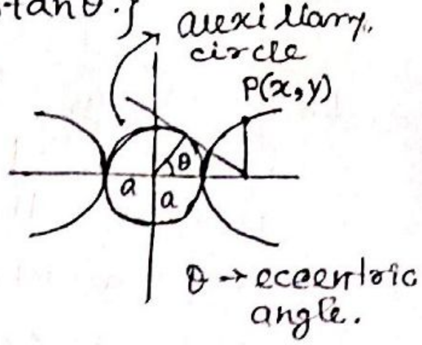
7. The difference of the focal radii of any point on the hyperbola is equal to the length of its transverse axis $(2a)$.

8. Rectangular Hyperbola: If $a=b$, the $x^2 - y^2 = a^2$ curve is rectangular or equilateral hyperbola. $(e = \sqrt{2})$

9. General eqn: $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$.

10. An ellipse and a hyperbola are confocal, conjugate axis of hyperbola = minor axis of ellipse, e_1, e_2 are their eccentricities. $\frac{1}{e_1^2} + \frac{1}{e_2^2} = 2$.

11. Parametric Coordinates: $\{x = a\sec\theta,\ y = b\tan\theta.\}$ $\hookrightarrow \theta \neq (2n+1)\frac{\pi}{2}$.



$$\begin{cases} x = a\left(\dfrac{e^{\theta}+e^{-\theta}}{2}\right) \\ y = b\left(\dfrac{e^{\theta}-e^{-\theta}}{2}\right). \end{cases}$$

equ^n of the chord ($a\sec\theta_1, b\tan\theta_1$; $a\sec\theta_2, b\tan\theta_2$)

$$\frac{x}{a}\cos\left(\frac{\theta_1-\theta_2}{2}\right) - \frac{y\sin}{b}\left(\frac{\theta_1+\theta_2}{2}\right) = \cos\left(\frac{\theta_1+\theta_2}{2}\right)$$

12. Position of a point with respect to the hyperbola: (x_1, y_1) outside, on; inside as

$$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 <, =, > 0.$$

13. Intersection of a line and a hyperbola:

i) Two distinct points intersection: $c^2 > a^2m^2 - b^2$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad y = mx + c$$

ii) Line touches hyperbola: $c^2 = a^2m^2 - b^2$

coordinates of point of contact.

$$\left(\pm\frac{a^2m}{\sqrt{a^2m^2-b^2}},\ \pm\frac{b^2}{\sqrt{a^2m^2-b^2}}\right).$$

14. If $(d,0)$ passes through the chord,

$$\frac{d}{a}\cos\frac{\theta_1-\theta_2}{2} = \cos\frac{\theta_1+\theta_2}{2} \Rightarrow \tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2} = \frac{a-d}{a+d}.$$

15. Tangent: Slope form: $y = mx \pm \sqrt{a^2m^2-b^2}$.

Point form: $\dfrac{xx_1}{a^2} - \dfrac{yy_1}{b^2} = 1$.

Parametric form: $\dfrac{x}{a}\sec\theta - \dfrac{y}{b}\tan\theta = 1$.

(at the point $a\sec\theta, b\tan\theta$).

16. Tangents at points $(a\sec\theta_1, b\tan\theta_1)$, $(a\sec\theta_2, b\tan\theta_2)$ intersect at the point –

$$\left(\frac{a\cos\left(\frac{\theta_1-\theta_2}{2}\right)}{\cos\left(\frac{\theta_1+\theta_2}{2}\right)},\ \frac{b\sin\left(\frac{\theta_1-\theta_2}{2}\right)}{\cancel{\sin}\cos\left(\frac{\theta_1+\theta_2}{2}\right)}\right)$$

17. Two tangents can be done from a point to a hyperbola.
Two tangents real-distinct $\frac{h^2}{a^2} - \frac{k^2}{b^2} < 1$.

18. Director Circle: $x^2 + y^2 = a^2 - b^2$.

19. Normal: Slope form: $y = mx \mp \frac{m(a^2+b^2)}{\sqrt{a^2 - b^2m^2}}$

at $\left(\pm \frac{a^2}{\sqrt{a^2-b^2m^2}}, \mp \frac{b^2m}{\sqrt{a^2-b^2m^2}} \right)$.

Point form: $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$.

Parametric form: $ax\cos\theta + by\cot\theta = a^2 + b^2$.

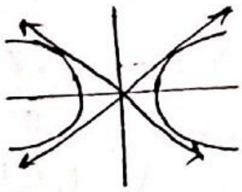
20. Four normals can be drawn from a point to hyperbola.

21. Conormal pt. - Points on hyperbola where four normals intersect.

* The sum of the eccentric angles of conormal point is an odd multiple of π.

22. Pair of tangents: $SS' = T^2$.

23. Asymptotes: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$. | $y = \pm \frac{b}{a}x$.



(2α) angle between the lines, then $\tan\alpha = b/a$, $\sec\alpha = e$.

24. Director Circle: $x^2 + y^2 = a^2 - b^2$, locus of the point of intersection of perpendicular tangents. It's real when $a^2 \geq b^2$, ie only if $1 < e \leq \sqrt{2}$.

25. Chord of contact of pair of tangents $T = 0$, Chord with mid-pt at (x_1, y_1) is $T = S_1$.

26. Two pt's with eccentric angle (α, β) joining " chord is focal if $\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{1+e}{1-e}$ or $\frac{1-e}{1+e}$

27. Rectangular hyperbola: $x^2 - y^2 = a^2$, $e = \sqrt{2}$, foci $(\pm\sqrt{2}a, 0)$, directrices $x = \pm\frac{a}{\sqrt{2}}$, LR = $2a$, asymptotes $x \pm y = 0$ (perpen L)

$xy = c^2$, asymptotes $xy = 0$, foci $(\sqrt{2}c, \sqrt{2}c)$ & $(-\sqrt{2}c, -\sqrt{2}c)$, directrices $\cancel{x = \pm\frac{a}{\sqrt{2}}}$ $x + y = \pm\sqrt{2}c$, LR = $\cancel{2a}$ $2\sqrt{2}c$,

vertices (c, c).

Parametric: $x = ct$, $y = \frac{c}{t}$, $t \neq 0$.

Tangent at $(x_1, y_1) \Rightarrow \frac{x}{x_1} + \frac{y}{y_1} = 2$.

Tangent at $t \rightarrow x + yt^2 = 2ct$.

Normal at $t \rightarrow t^2x - y = c\left(t^3 - \frac{1}{t}\right)$.

Normal at t meets hyperbola again at t' if $t^3t' + 1 = 0$.

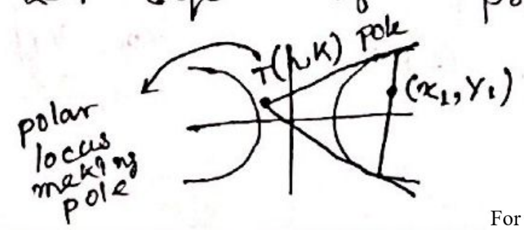
If a circle cuts $xy = c^2$ at 4 pt's then $x_1x_2x_3x_4 = y_1y_2y_3y_4 = c^4$.

The orthocentre of a triangle with vertices $\left(ct_i, \frac{c}{t_i}\right)$ inscribed in $xy = c^2$ lies on the curve, whose coordinates are $\left(-\frac{c}{t_1t_2t_3}, -ct_1t_2t_3\right)$

28. Eqn of chord of hyperbola bisected at (x_1, y_1)

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1. \Rightarrow T = S'$$

29. Eqn of polar through pt. (x_1, y_1).



$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$$

30. Conjugate points: Two points are said to be conjugate points with respect to a hyperbola, if each lies on the polar of the other. If (x_1, y_1) & (x_2, y_2) are conjugate pts,

$$\frac{x_1 x_2}{a^2} - \frac{y_1 y_2}{b^2} = 1.$$

Conjugate lines: If each line passes through the pole of the other.

31. Diameter: locus of the mid pts of parallel chords

Eqn: $y = \frac{b^2}{a^2 m} x$ [m is the slope of chords].

Conjugate Diameter: 2 diameters are conjugate if each bisects the chords parallel to the other.