Handwritten Notes

On

Differential Equations

Differential Equations.

* **Differential Equation:** An equation involving independent variable, dependent and the differential coefficients of the dependent variable.

* **Order & Degree of Differential Equ^n:**

The order of the highest differential coefficient appearing in the differential equ^n is called order of differential equ^n.

The exponent of the highest differential coefficient, when the differential equ^n is a polynomial on all the differential coefficients, is known as the degree of the differential equation.

* **Formation of Differential Equations:**

a) Differentiate the given equ^n w.r.t. the independent variable as many times as the number of arbitlary constants in it.

b) Eliminate the arbitlary constants.

* The order of the differential equ^n will be equal to number of independent parameters and is not equal to the number of all the parameters in the family of curves.

* Solution of a differential equ^n:

a) A general Sol^n or an integral: A relation between the variables (not involving the derivatives) which contains the same number of arbitary constants as the order of equation.

b) Particular solution or particular integral: Obtained from the general solution by assigning particular values to the arbitary constant in the general solution.

* Diff. equ^n of 1st Order - 1st Degree: $\frac{dy}{dx} = f(x,y)$.

- Equations in which the variables are seperable: If $\frac{dy}{dx} = f(x,y)$ can be expressed in the form $f(x)\,dx = g(y)\,dy$. By integrating this, solution-

$$\int f(x)\,dx = \int g(y)\,dy + c.$$

- Reducible to the separable variable type: $\frac{dy}{dx} = f(ax+by+c)$ is solved by putting $ax+by+c = t$.

- Homogeneous diff. equ^n:

i) $P(x,y)\,dx + Q(x,y)\,dy = 0$ is called homogeneous, if P & Q are homogeneous functions of the same degree in x & y. Reducible to $y' = f(y/x)$, substitute $y = xu$, u is unknown function. The equ^n is transformed to an equ^n with variable separables.

ii) $\frac{dy}{dx} = f\left(\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}\right)$, $a_1b_2 - a_2b_1 \neq 0$.

then substitute $x = u+h$, $y = v+k$

if $a_1b_2 - a_2b_1 = 0$, $u = a_1x + b_1y$. transforms into a variable separable form.

* $P(x,y)$ function is homogeneous of degree n, if for any real t. $P(tx,ty) = t^n(P(x,y))$.

Differential Equations

* A diff. equ^n of the form $\frac{dy}{dx} = f(x,y)$ is homogeneous, if $f(x,y)$ is a homogeneous function of degree zero ie,

$f(tx, ty) = t^0 . f(x,y)$

* Exact differential equ^n:

$M(x,y)\,dx + N(x,y)\,dy = 0$ is exact if its LH expression is the exact differential of some function $u(x,y)$.

$$du = M dx + N dy.$$

Then sol^n is $u(x,y) = c$.

- The sufficient condition for the diff. equ^n $M dx + N dy = 0$ to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

The solution of $M dx + N dy = 0$ is

$$\int_{y\text{-constant}} M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = c$$

provided $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

* Solution by inspection:

1) $d(xy) = x\,dy + y\,dx$
2) $d(x/y) = \frac{x\,dx - x\,dy}{y^2}$
3) $d(y/x) = \frac{x\,dy - y\,dx}{x^2}$
4) $d(x^2/y) = \frac{2xy\,dx - x^2\,dy}{y^2}$
5) $d(y^2/x) = \frac{2xy\,dy - y^2\,dx}{x^2}$
6) $d(x^2/y^2) = \frac{2xy^2\,dx - 2x^2y\,dy}{y^4}$
7) $d(y^2/x^2) = \frac{2x^2y\,dy - 2xy^2\,dx}{x^4}$
8) $d\left(\tan^{-1}\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{x^2+y^2} = \frac{d(x/y)}{1+(x/y)^2}$

9) $d\left(\tan^{-1}\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2+y^2} = \frac{\frac{x\,dy - y\,dx}{x^2}}{1+y^2/x^2} = \frac{d(y/x)}{1+(y/x)^2}$

10) $d[\ln(xy)] = \frac{x\,dy + y\,dx}{xy}$ 11. $d\left[\ln\left(\frac{x}{y}\right)\right] = \frac{y\,dx - x\,dy}{xy}$

12) $d\left[\ln\left(y/x\right)\right] = \frac{x\,dy - y\,dx}{xy}$ 13. $d\left[\frac{1}{2}\ln(x^2+y^2)\right] = \frac{x\,dx + y\,dy}{x^2+y^2}$

14) $d\left(-\frac{1}{xy}\right) = \frac{x\,dy + y\,dx}{x^2y^2}$ 15) $d\left(\frac{e^x}{y}\right) = \frac{ye^x\,dx - e^x\,dy}{y^2}$

16) $d\left(\frac{e^y}{x}\right) = \frac{xe^y\,dy - e^y\,dx}{x^2}$ 17) $d(x^m y^n) = x^{m-1}y^{n-1}(my\,dx + nx\,dy)$.

18) $\frac{d[f(x,y)]^{1-n}}{1-n} = \frac{f'(x,y)}{(f(x,y))^n}$.

19)* If $x = r\cos\theta$, $y = r\sin\theta$.

i) $x\,dx + y\,dy = r\,dr$ ii) $x\,dy - y\,dx = r^2\,d\theta$

iii) $dx^2 + dy^2 = dr^2 + r^2\,d\theta^2$

* 1st Order Linear diff. equ^n: $\frac{dy}{dx} + P(x)\cdot y = Q(x)$

Solution: $y e^{\int P\,dx} = \int Q e^{\int P\,dx}\,dx + c$

$e^{\int P\,dx} = IF$ - Integrating factor.

* Bernoulli's Equ^n: $\frac{dy}{dx} + Py = Qy^n$

dividing $y^n \rightarrow y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$. —(1)

$y^{1-n} = z \Rightarrow (1-n)\,y^{-n}\frac{dy}{dx} = \frac{dz}{dx}$.

(1) $\Rightarrow \frac{dz}{dx} + (1-n)Pz = (1-n)Q$.

$IF = e^{\int(1-n)P\,dx}$

Solution is

$z e^{\int(1-n)P\,dx} = \int\left\{(1-n)Q\cdot e^{\int(1-n)P\,dx}\right\}dx$.

* Orthogonal Trajectory: Any curve, which cuts every member of a given family of curves at right angles, is called an orthogonal trajectory of the family.

• Procedure for finding OT:

i) Let $f(x, y, c) = 0$ is the equation of family.

ii) Differentiate $f = 0$, wrt x. & eliminate c.

iii) Substitute $-\frac{dx}{dy}$ for $\frac{dy}{dx}$. That is the diff. equ^n of OT. Now, solve it to get OT.