Handwritten Notes

On

Binomial Theorem

# Binomial Theorem.



* Binomial Expression: An algebraic expression consisting of two terms with +ve or -ve sign between them is called a binomial expression.

* Binomial theorem for positive integral index:

$$(x+y)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} y + \dots + {}^nC_r x^{n-r} y^r + \dots + {}^nC_n x^0 y^n.$$

$$(x+y)^n = \sum_{r=0}^{n} {}^nC_r x^{n-r} y^r.$$

- $$(x-y)^n = \sum_{r=0}^{n} (-1)^r \, {}^nC_r x^{n-r} y^r.$$

- $$(x+y)^n + (x-y)^n = 2\{x^n + {}^nC_2 x^{n-2} y^2 + {}^nC_4 x^{n-4} y^4 + \dots\}$$
$$= 2\{\text{sum of terms at odd places}\}.$$

The last term is ${}^nC_n y^n$ or ${}^nC_{n-1} x y^{n-1}$ according as $n$ is even or odd.

- $$(x+y)^n - (x-y)^n = 2\{{}^nC_1 x^{n-1} y + {}^nC_3 x^{n-3} y^3 + \dots\}$$
$$= 2\{\text{Sum of terms at even places}\}.$$

The last term is ${}^nC_{n-1} x y^{n-1}$ or ${}^nC_n y^n$ according as $n$ is even or odd.

- $(1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r$
- $(1-x)^n = \sum_{r=0}^{n} (-1)^r \, {}^nC_r x^r$

* Properties of Binomial Expansion $(x+y)^n$:

i) There are $n+1$ terms in expansion.

ii) ${}^nC_r = {}^nC_{n-r}$ ie the coefficients of the terms equidistant from the beginning & the end in a binomial expansion are equal.

iii) For each term sum of indices of $x$ & $y$ is equal to $n$.

* General Term: $T_{r+1} = (r+1)^{th}$ term $= {}^nC_r\, x^{n-r} y^r$
* $p^{th}$ term from end: $p^{th}$ term from the end in $(x+y)^n$ is $(n-p+2)^{th}$ term from the beginning.
* Middle Terms:

a) When $n$ is even: The $\left(\frac{n}{2}+1\right)^{th}$ term is the only one middle term.

$$T_{\frac{n}{2}+1} = {}^nC_{n/2}\, x^{n/2} y^{n/2}$$

b) When $n$ is odd: $\left(\frac{n+1}{2}\right)^{th}$ & $\left(\frac{n+3}{2}\right)^{th}$ terms are two middle terms.

* Greatest term: $\dfrac{T_{r+1}}{T_r} = \dfrac{n-r+1}{r} x$.

Let numerically $T_{r+1}$ be the greatest term.

$$\frac{T_{r+1}}{T_r} \geq 1. \Rightarrow r \leq \frac{(n+1)}{(1+|x|)}|x|$$

$n, x$ substituting, $r \leq m+f$ or $r \leq m$.

$m$ is +ve integer & $f$ is fraction $(0<f<1)$

If $r \leq m+f$, $T_{m+1}$ is the greatest.

If $r \leq m$, $T_m$ & $T_{m+1}$ are greatest, & both are equal.

* Finding greatest term in $(x+y)^n$.

Since, $(x+y)^n = x^n\left(1+\frac{y}{x}\right)^n$

Then, find the greatest term in $\left(1+\frac{y}{x}\right)^n$.

* Greatest Coefficient: If $n$ is even, then ${}^nC_{n/2}$.

If $n$ is odd then ${}^nC_{(n-1)/2}$ & ${}^nC_{(n+1)/2}$.

* Properties of Binomial Coefficients:

In the expansion $(1+x)^n$, the coefficients are ${}^nC_0, {}^nC_1, \ldots, {}^nC_r, \ldots, {}^nC_n$ (by $C_0, C_1, C_2, \ldots, C_n$).

1. The sum of the binomial coefficients in the expansion of $(1+x)^n$ is $2^n$. $C_0 + C_1 + \cdots + C_n = 2^n$

2. The sum of the coefficients of odd terms in the expansion $(1+x)^n$ is equal to the sum of the coefficients of the even terms & both is equal to $2^{n-1}$

3. $C_0^2 + C_1^2 + C_2^2 + \cdots + C_n^2 = {}^{2n}C_n = \dfrac{2n!}{(n!)^2}$

4. $C_0 C_r + C_1 C_{r+1} + C_2 C_{r+2} + \cdots + C_{n-r} C_n = {}^{2n}C_{n-r} = \dfrac{2n!}{(n+r)!\,(n-r)!}$

* If $(\sqrt{A}+B)^n = I+f$ where $I$ & $n$ are +ve integers, $n$ being odd & $0 \le f < 1$, then show that $(I+f)\,f = k^n$ where $A-B^2 = k > 0$ & $\sqrt{A} - B < 1$.

When $n$ is even, $(I+f)(1-f) = k^n$.

* Multinomial Theorem: If $n$ is +ve integer & $a_1, a_2, a_3, \ldots, a_m \in C$ then

$$(a_1 + a_2 + \cdots + a_m)^n = \sum \frac{n!}{n_1!\, n_2! \cdots n_m!} a_1^{n_1} a_2^{n_2} \cdots a_m^{n_m}$$

where $n_1, n_2, \ldots, n_m$ are all non-negative integers subject to the condition $n_1 + n_2 + n_3 + \cdots + n_m = n$

- The coefficient of $a_1^{n_1} a_2^{n_2} \dots a_m^{n_m}$ in the expansion of $(a_1 + a_2 + \dots + a_m)^n$ is $\frac{n!}{n_1! \, n_2! \dots n_m!}$.

- Greatest coefficient of $(a_1 + a_2 + \dots + a_m)^n$ is

$$\frac{n!}{(q!)^{m-r}((q+1)!)^r}$$ , where $q$ is the quotient & $r$ is the remainder when $n$ is divided by $m$.

- If $n$ is a +ve integer & $a_1, a_2, \dots, a_m \in C$ then coefficient of $x^r$ in the expansion of $(a_1 + a_2 x + a_3 x^2 + \dots + a_m x^{m-1})^n$ is

$$\sum' \frac{n!}{n_1! \, n_2! \dots n_m!} a_1^{n_1} a_2^{n_1} \dots a_m^{n_m}$$

where $n_1, n_2, \dots, n_m$ are all non negative integers subject to the condition

$n_1 + n_2 + n_3 + \dots + n_m = n$ &

$n_2 + 2n_3 + 3n_4 + \dots + (m-1) n_m = r$.

- Number of distinct or dissimilar terms in the expansion $(a_1 + a_2 + \dots + a_m)^n$ is ${}^{n+m-1}C_{m-1}$.

* Use of differentiation: This method applied only when the numericals occur as the product of the binomial coefficients.

Solution- i) If last term of the series leaving the plus or minus sign be $m$, then divide the $m$ by $n$ if $q$ be the quotient & $r$ be the remainder. Then replace $x$ by $x^q$ in given series & multiplying both sides of the expression by $x^r$. ii) Differentiate both sides wrt $x$ & put $x = 1$ or $-1$ or $i$ or etc according to the given series. iii) If product of two numericals (or square of numericals) or three numericals (or cube of numericals) then differentiate twice or thrice.

* Use of Integration: This method is only applied when the numericals occur as the denominator of the binomial coefficient.

Solution: If $(1+x)^n = C_0 + C_1 x + \dots + C_n x^n$ then integrate both sides between suitable limits which gives required series. (i) If sum contains $C_0, C_1, C_2, \dots, C_n$ are all +ve signs, then integrate between 0 to 1.
(ii) If sum contains alternate signs then between (-1 to 0). (iii) If sum contains odd coefficients $(C_0, C_2, C_4)$ then between -1 to +1.
(iv) If sum contains even coefficients $(C_1, C_3, \dots)$ then subtracting (ii) from (i) then dividing by 2.
(v) If in denominator of binomial coefficient product of two numericals then integrate two times first times taken limits between 0 to $x$ & second time, take suitable limits.

* If $(1+x)^n = C_0 + C_1 x + C_2 x + \dots + C_n x^n$ then

$$C_0C_1 + C_1C_2 + \dots + C_{n-1}C_n = \frac{n!}{(n-1)!(n+1)!}$$

$$C_0C_2 + C_1C_3 + \dots + C_{n-2}C_n = \frac{2n!}{(n-2)!(n+2)!}$$

* If $n > 6$, then $\left(\frac{n}{3}\right)^n < n! < \left(\frac{n}{2}\right)^n$

# Binomial Theorem for any Rational Index.

* Whatever be the value of $n$, $(1+x)^n =$

$$1+\frac{n}{1!}x+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\dots+\frac{n(n-1)(n-2)\dots(n-r+1)}{r!}x^r+\dots \text{to } \infty$$

where $|x|<1$ i.e. $-1<x<1$.

* If $n$ is not a +ve integer, we should write $(x+a)^n$ as $x^n\left(1+\frac{a}{x}\right)^n$, if $x>a$ and as $a^n\left(1+\frac{x}{a}\right)^n$, if $x<a$.

* When $x$ is so small $[\,|x|<1\,]$ then $(1+x)^n \approx 1+nx$.

* $(r+1)^{th}$ term, $T_{r+1} = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}x^r$.

* $$(1-x)^n = 1-\frac{n}{1!}x+\frac{(n)(n-1)}{2!}x^2-\frac{n(n-1)(n-2)}{3!}x^3+\dots+(-1)^r\frac{n(n-1)(n-2)\dots(n-r+1)}{r!}x^r+\dots \text{to } \infty$$

* $(1+x)^{-1} = 1-x+x^2-x^3+x^4-\dots\infty$

$(1-x)^{-1} = 1+x+x^2+x^3+x^4+\dots\infty$

$(1+x)^{-2} = 1-2x+3x^2-4x^3+\dots\infty$

$(1-x)^{-2} = 1+2x+3x^2+4x^3+\dots\infty$